AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

2006 IMO Team Selection School, April 10-19, Shore School, Sydney

Preparation Problems for April 2006

A. Easier Problems

1. In the world soccer cup, the teams are divided into groups of four. Each team plays one game with every other team in its group. A win gives 3 points, a draw 1 point and a loss 0 points. From each group, two teams advance so that each advancing team gets at least as many points as each non-advancing team.

   (a) What is the smallest possible score of an advancing team?

   (b) What is the largest possible score of a non-advancing team?

2. In triangle ABC let D, E be the midpoints of AB and AC, respectively. Prove that the intersection point of the bisectors of angles BDE and CED lies on BC if and only if the length of BC is equal to the arithmetic mean of the lengths of AB and AC.

3. Let $n > 3$ be a positive integer. Consider n sets, each having two elements. It is known that each pair of sets has exactly one element in common. Prove that all of the sets have exactly one element in common.

4. Let $ABCD$ be a quadrilateral and let O be a point in the same plane as $ABCD$. Let K, L, M and N be the circumcentres of triangles AOB, BOC, COD and DOA, respectively. Prove that there exists exactly one point O in the plane such that $KLMN$ is a parallelogram.

5. Let $ABCD$ be a rectangle. Let P be the midpoint of AB and let Q be the point on PD such that CQ is perpendicular to PD. Prove that triangle BQC is isosceles.

6. Can you find a positive integer n such that $4^{995} + 4^{1500} + 4^n$ is a perfect square?

7. Two circles C_1 and C_2 with centres O_1 and O_2, respectively, are tangent externally at P. On their common tangent at P, point A is chosen, rays drawn from which are tangent to the circles C_1 and C_2 at points P_1 and P_2 both different from P. It is known that $\angle P_1AP_2 = 120°$ and angles $\angle P_1AP$ and $\angle P_2AP$ are both acute. Rays AP_1 and AP_2 intersect line O_1O_2 at points G_1 and G_2, respectively. The second intersection between ray AO_1 and C_1 is H_1, the second intersection between ray AO_2 and C_2 is H_2. Lines G_1H_1 and AP intersect at K. Prove that if G_1K is tangent to circle C_1, then the line G_2K is tangent to circle C_2 with tangency point H_2.

8. A postal service of some country uses carriers to transport the mail. Each carrier's task is to bring the mail from one city to a neighbouring city. It is known that it is possible to send mail from any city to the capital P. For any two cities A and B, call B *more important than* A, if every possible route of mail from A to the capital P goes through B.

   (a) Prove that for any three different cities A, B and C, if B is more important than A and C is more important than B, then C is more important than A.

   (b) Prove that for any three different cities A, B and C, if both B and C are more important than A, then either C is more important than B or B is more important than C.

9. Consider a convex n-gon in the plane with n being odd. Prove that if it is possible to find a point P in the plane from which all the sides of the n-gon are viewed at equal angles, then this point is unique. (We say that segment AB is *viewed at angle γ from point P* if and only if $\angle APB = \gamma$.)

10. Represent the number $\sqrt[3]{1342\sqrt{167} + 2005}$ using only integers and the $+, -, \times, \div, \sqrt{\ }, (,)$ symbols.

11. Positive integers a and b are chosen such that $\frac{a}{b}\sqrt{a^2 + b^2}$ is an integer.

    (a) Is it necessarily true that every prime factor of b is a prime factor of a? (b) Is it necessarily true that $b \le a$?

12. Let ABC be an obtuse triangle inscribed in a circle of radius 1. Prove that triangle ABC can be completely covered by an isosceles right angled triangle with hypotenuse $\sqrt{2} + 1$.

13. Joel and Kyle play the following game on an $m \times n$ square array. Each player on his turn is permitted to colour in two squares having a common side which have not already been coloured in. Joel starts and moves are made in turns. A player who cannot move loses. Who wins the game if

    (a) $m = 3$ and $n = 3$? (b) $m = 2004$ and $n = 2004$? (c) $m = 2004$ and $n = 2005$?

14. Relatively prime positive integers a and b are chosen in such a way that $\frac{a+b}{a-b}$ is also a positive integer. Prove that at least one of the numbers $ab + 1$ and $4ab + 1$ is a perfect square.

15. Let a, b and n be integers such that $a + b$ is divisible by n and $a^2 + b^2$ is divisible by n^2. Prove that $a^m + b^m$ is divisible by n^m for all positive integers m.

B. Medium Problems

16. Does there exist an integer $n \geq 2$ such that $2^{2^n-1} - 7$ is not a perfect square?

17. Solve the following system of equations in integers.

$$\begin{array}{rcl} x(y+z+1) & = & y^2+z^2-5 \\ y(z+x+1) & = & z^2+x^2-5 \\ z(x+y+1) & = & x^2+y^2-5 \end{array}$$

18. Let $a \geq 2$ be a positive integer. Prove that if $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_l\}$ are different sets of positive integers, then $(a^{i_1}+1)(a^{i_2}+1)\ldots(a^{i_k}+1) \neq (a^{j_1}+1)(a^{j_2}+1)\ldots(a^{j_l}+1)$.

19. Prove that if $ABCD$ is a convex quadrilateral $(AB+CD)^2 + (BC+DA)^2 \geq (AC+BD)^2$.

20. Let p be a prime number. Find all integers k such that $\sqrt{k^2 - pk}$ is also an integer.

21. Prove that $\frac{a+b}{c^2} + \frac{b+c}{a^2} + \frac{c+a}{b^2} \geq 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$ for any positive real numbers a, b, c.

22. Find the value of $abcd$ where a, b, c, d are real numbers satisfying:

$$a = \sqrt{45 - \sqrt{21 - a}}, \quad b = \sqrt{45 - \sqrt{21 - b}}, \quad c = \sqrt{45 - \sqrt{21 + c}}, \quad d = \sqrt{45 - \sqrt{21 + d}}.$$

23. Determine for which natural numbers k, the system of inequalities $k(k-2) \leq \left(k + \frac{1}{k}\right) x \leq k^2(k+3)$ with unknown x and parameter k has exactly $(k+1)^2$ solutions in the domain of the integers.

24. Find all pairs of positive integers x, y such that the sum of their arithmetic and geometric means equals 40.

25. Let $a, b, c > 0$ with $abc = 1$. Prove that $\dfrac{a}{(a+1)(b+1)} + \dfrac{b}{(b+1)(c+1)} + \dfrac{c}{(c+1)(a+1)} \geq \dfrac{3}{4}$.

26. Find all pairs of natural numbers such that the last digit of their sum is a 3, their difference is a prime, and their product is a perfect square.

27. Let ABC be a triangle with $\angle BAC = 90^\circ$. Let D be the intersection of the angle bisector of $\angle BAC$ with the line BC. Let I_a be the centre of the excircle of triangle ABC with respect to A. Prove that $\frac{AD}{DI_a} \leq \sqrt{2} - 1$.

28. Find all functions $f : \mathbb{N}^+ \to \mathbb{N}^+$ such that for all $m, n \in \mathbb{N}^+$ we have $f(m) + f(n) | m + n$.

29. A deck of 32 cards has 2 different jokers each of which is numbered 0, 10 red cards numbered 1 through 10, 10 blue cards numbered 1 through 10 and 10 green cards numbered 1 through 10. One chooses a hand of cards from the deck. If a card in the hand is numbered k, then the value of the card is 2^k and the value of the hand is the sum of the values of the cards in the hand. Determine the number of hands of value 2005.

30. Find the minimum value of $u+v+w$ where u, v and w are positive real numbers with $u\sqrt{vw} + v\sqrt{wu} + w\sqrt{uv} \geq 1$.

31. 2005 positive integers whose sum equals 7022 are written on a circle. Prove that it is possible to find two pairs of neighbouring numbers such that the sum of the numbers in each pair is at least 8.

C. Difficult Problems

32. Let a, b, c be positive real numbers satisfying $(a+b)(b+c)(c+a) = 1$. Show that $ab + bc + ca \leq \frac{3}{4}$.

33. In the plane, a line λ and two circles C_1 and C_2 of different radii are given such that λ touches both circles at point P. Point $M \neq P$ on λ is chosen so that $\angle Q_1 M Q_2$ is as large as possible where Q_1 and Q_2 are the tangency points of the tangent lines drawn from M to C_1 and C_2, respectively, differing from λ. Find $\angle PMQ_1 + \angle PMQ_2$.

34. Prove that not all the roots of $P(x) = x^n - 5x^{n-1} + 12x^{n-2} - 15x^{n-3} + a_{n-4}x^{n-4} + \cdots + a_0$ are positive real numbers.

35. The diagonals of trapezium $ABCD$ with bases AB and CD are mutually perpendicular and intersect at O. Let M and N be points on the rays OA and OB respectively, such that $\angle ANC = \angle BMD = 90^\circ$. Let E be the midpoint of MN. Prove that: (a) triangles OMN and OBA are similar; (b) $OE \perp AB$.

36. Find all pairs (a, b) of real numbers such that the roots of polynomials $6x^2 - 24x - 4a$ and $x^3 + ax^2 + bx - 8$ are all non-negative real numbers.

37. Let A and B be two finite sets of distinct points in the plane. For every four distinct points of $A \cup B$ there is a line such that it separates the points of A and B among these four. Prove that there is a line which separates A and B.

38. Let $ABCD$ be a tetrahedron. Let H_a, H_b, H_c and H_d be the orthocentres of triangles BCD, ACD, ABD and ABC, respectively. Prove that AH_a, BH_B, CH_c and DH_d are concurrent if and only if $AB^2 + CD^2 = AC^2 + BD^2 = AD^2 + BC^2$.

39. Solve the following equation in prime numbers $p^3 = p^2 + q^2 + r^2$.